Multipartite Moore Digraphs *

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Abstract

We derive some Moore-like bounds for multipartite digraphs, which extend those of bipartite digraphs, under the assumption that every vertex of a given partite set is adjacent to the same number δ of vertices in each of the other independent sets. We determine when a Moore multipartite digraph is weakly distance-regular. Within this framework, some necessary conditions for the existence of a Moore r-partite digraph with interpartite outdegree $\delta > 1$ and diameter $k = 2m$ are obtained. In the case $\delta = 1$, which corresponds to almost Moore digraphs, a necessary condition in terms of the permutation cycle structure is derived. Additionally, we present some constructions of dense multipartite digraphs of diameter two that are vertex-transitive.

Key words. Multipartite digraph; Moore digraph; Weakly distance-regular digraph; Degree/diameter problem; Eigenvalues.

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1 Introduction

A fundamental area in the study of graphs is concerned with the question of how 'large' a directed graph (digraph) can be in terms of the number of arcs and/or vertices, given some constraints. Dense digraphs are interesting because of their possible

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application to model good interconnection networks, and also by themselves because high density usually implies a large amount of structure. One of the prominent problems in this area is the well known *degree/diameter problem* which is to determine, for each d and k, the largest order $n_{d,k}$ of a digraph of maximum outdegree d and diameter at most k. While this problem is in general wide open and considered to be very difficult, one possible way forward is to consider this problem for restricted classes of digraphs. For example, the first and second author and their collaborators [14, 13] previously restricted their attention to bipartite digraphs. In this paper we will consider the degree/diameter problem for multipartite digraphs.

For general digraphs, the optimum situation would be to have digraphs with order attaining the so-called *Moore bound*

$$M(d,k) := 1 + d + \cdots + d^k = \frac{d^{k+1}-1}{d-1} \qquad (d > 1) \tag{1}$$

but unfortunately such digraphs are very scarce. In fact they only exist for the trivial values $d = 1$ (the directed cycle C^*_{k+1}) or $k = 1$ (the complete symmetric digraph K^*_{d+1}); see [22, 6].

Terminology and notation

Let $G = (V, E)$ denote a digraph with vertex set V, arc set E, and distance function $\mathrm{dist}(\cdot,\cdot)$. If $U \subset V$, the distance from $v \in V$ to the set U is defined as expected, that is, $\mathrm{dist}(v, U) := \min_{u\in U} \mathrm{dist}(v, u)$. Let G have diameter k. Then, for a given integer l, $0 \le l \le k$, and a vertex u, we denote by $\Gamma^+_l(u)$ and $N^+_l(u)$ the sets of vertices at distance l and $\le l$, respectively, from u. Similarly, $\Gamma^-_l(u)$ and $N^-_l(u)$ stand for the sets of vertices at distance l and $\le l$, respectively, to u. In particular, $\Gamma^+(u) := \Gamma^+_1(u)$ and $\Gamma^-(u) := \Gamma^-_1(u)$ are the sets of *outneighbours* and *inneighbours* of u, and their cardinalities are the *outdegree* $\delta^+(u)$ and *indegree* $\delta^-(u)$ of vertex u, respectively.

A digraph $G = (V, E)$ is r*-partite* (or, generically, *multipartite*) if, for some integer $r > 1$, its vertex set admits a partition into r parts, $V = V_1 \cup V_2 \cup \cdots \cup V_r$, such that every arc $(u, v) \in E$ is of the form $u \in V_i$ and $v \in V_j$ with $1 \le i, j \le r$ and $j \ne i$. In other words, there are no arcs between vertices within the same (*independent* or *partite*) set V_i, $1 \le i \le r$. For symmetry reasons, we restrict our study to some particular classes of multipartite digraphs, which we call equipartite, equiregular and equioutregular. An r-partite digraph on n vertices is said to be (r-)*equipartite* if its independent sets have all equal cardinality (n/r). A multipartite digraph is called (δ-)*equiregular* if it is d-regular, with degree $d = (r-1)\delta$ for some $\delta \ge 1$, and each vertex $u \in V_i$ has exactly δ inneighbours and outneighbours in each of the parts V_j, $j \ne i$. Moreover, if this regularity condition holds only for the outdegrees, we say that G is (δ-)*equioutregular*.

Note that, since any r-partite digraph G of order n is also trivially r'-partite for any $r' \ge r$, where $r' \le n$, we can consider only the values $2 \le r \le \chi$, where χ is the chromatic number of the underlying graph $U(G)$. But, if $U(G) \notin \{K_n, C_{2p+1}\}$ has maximum degree Δ then $\chi \leq \Delta$ (Brooks' theorem, see [7]). Thus, if our digraphs are d-regular, it suffices to consider the cases $r \leq 2d$.

Local multiplicities and walks

Let G be an (undirected) graph with adjacency matrix $\boldsymbol{A}$ and eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. For each eigenvalue λ_i, $0 \leq i \leq d$, let $\boldsymbol{E}_i$ denote the so-called (*principal*) *idempotent* of $\boldsymbol{A}$; that is, the matrix which represents the orthogonal projection onto the eigenspace $\mathcal{E}_i$ associated to λ_i. Accordingly, such matrices satisfy $\boldsymbol{E}_i\boldsymbol{E}_j = \delta_{ij}\boldsymbol{E}_i$, and $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i)\boldsymbol{E}_i$, for any polynomial $p \in \mathbb{R}[x]$. In particular, taking $p = x^l$, the l-th power of $\boldsymbol{A}$ can be expressed as a linear combination of the idempotents $\boldsymbol{E}_i$: $\boldsymbol{A}^l = \sum_{i=0}^{d} \lambda_i^l \boldsymbol{E}_i$, and hence the number of l-walks between two vertices u and v of G is given by

$$(\boldsymbol{A}^l)_{uv} = \sum_{i=0}^{d} \lambda_i^l (\boldsymbol{E}_i)_{uv}. \tag{2}$$

The numbers $(\boldsymbol{E}_i)_{uv}$ are called in [12] the *crossed* (uv-)*local multiplicities* of λ_i, and are denoted by $m_{uv}(\lambda_i)$. In the case $u = v$ it turns out that $m_{uu}(\lambda_i)$, called simply the (u-)*local multiplicities*, play a role, in a local context, similar to the role of the standard multiplicities $m(\lambda_i)$ of the eigenvalues. In fact, if G is a distance-regular graph on n vertices with distance polynomials p_k, $0 \leq k \leq d$, then the uv-local multiplicities can be computed using the formulas

$$m_{uv}(\lambda_i) = \frac{m(\lambda_i)}{n} \frac{p_k(\lambda_i)}{p_k(\lambda_0)}, \tag{3}$$

provided that $\operatorname{dist}(u, v) = k$; see [11]. In particular, for $k = 0$ the above gives $m_{uu}(\lambda_i) = m(\lambda_i)/n$ for any eigenvalue λ_i and vertex u. Then, by (2), the graph has the property that the number of closed l-walks rooted at a given vertex u does not depend on the chosen vertex. Graphs satisfying such a property have been called *walk regular* [19] or *spectrally regular* [12].

2 Some Moore-like bounds

The main result of this section is to derive some Moore-like bounds for multipartite digraphs, and compare them with each other. Note first that, if G is an equipartite digraph with maximum outdegree d and r independent sets, a straightforward upper bound on its number of vertices is given by the following result.

Proposition 2.1. *Let G be an r-equipartite digraph with maximum outdegree d and diameter k. Then*

$$n \leq \overline{M}_r(d, k) := r \left\lfloor \frac{M(d, k)}{r} \right\rfloor \tag{4}$$

where $M(d, k)$ is the (ordinary) Moore bound (1), and $\lfloor \cdot \rfloor$ denotes integer part.

Thus, depending on the divisibility of $M(d,k)$ by r, the difference between $\overline{M}_r(d,k)$ and $M(d,k)$ can be (obviously) as much as $r-1$.

In spite of its simplicity, the above bound has the advantatge of applying to any combination of values of d, k and r. However, to proceed further with our study, we next consider the more restrictive case $r \le d+1$, and assume that our digraphs are equioutregular. Then, in this case, each vertex of a given partite set is adjacent to vertices of all the other partite sets. In order to avoid trivial cases, henceforth we will not consider the case of degree $d=1$ (directed cycles).

Proposition 2.2. *Let G be a δ-equioutregular r-partite digraph with outdegree $d = \delta(r-1) > 1$ and diameter k. Then its number of vertices n satisfies the following Moore-like bounds:*

(a) *For odd diameter, $k = 2m+1$,*

$$\begin{aligned} n \le M_r(d,k) &:= M(d,k) + (r-1)M(-\delta,k) \\ &= \frac{d^{k+1}-1}{d-1} - (r-1)\frac{\delta^{k+1}-1}{\delta+1}. \end{aligned} \tag{5}$$

(b) *For even diameter, $k = 2m$,*

$$\begin{aligned} n \le M_r(d,k) &:= M(d,k) - M(-\delta,k) \\ &= \frac{d^{k+1}-1}{d-1} - \frac{\delta^{k+1}+1}{\delta+1}. \end{aligned} \tag{6}$$

Proof. The complete graph K_r, with vertices $1,2,\ldots,r$ and adjacency matrix $\boldsymbol{B} = \boldsymbol{J} - \boldsymbol{I}$, is trivially distance-regular with distance polynomials $p_0 = 1$ and $p_1 = x$. Then any vertex $i \in V(K_r)$ has local eigenvalues $\lambda_0 = r-1$ and $\lambda_1 = -1$; that is, those of K_r. From (3), the local multiplicities of the eigenvalues are $m_i(\lambda_0) = 1/r$ and $m_i(\lambda_1) = (r-1)/r$. Hence, the number of closed (i,i)-walks of length $l \ge 0$ is

$$(\boldsymbol{B}^l)_{ii} = \frac{1}{r}\operatorname{tr}\boldsymbol{B}^l = \frac{1}{r}[(r-1)^l + (r-1)(-1)^l].$$

Thus, for any given vertex $u \in V_i$, $1 \le i \le r$, the number of vertices of the same partite set V_i which are at distance $\le k$ from u; that is, $\sigma_k(i,i) := |N_k^+(u) \cap V_i|$, satisfies the upper bound

$$\begin{aligned} \sigma_k(i,i) \le \sum_{l=0}^{k} \delta^l (\boldsymbol{B}^l)_{ii} &= \frac{1}{r}\left\{ \sum_{l=0}^{k} \delta^l [(r-1)^l + (r-1)(-1)^l] \right\} \\ &= \frac{1}{r}\left\{ \sum_{l=0}^{k} d^l + (r-1)\sum_{l=0}^{k}(-\delta)^l \right\}. \end{aligned} \tag{7}$$

Moreover, if i,j are adjacent vertices of K_r, the (crossed) ij-local multiplicities of the eigenvalues $\lambda_0 = r-1$ and $\lambda_1 = -1$ are given by $m_{ij}(\lambda_0) = 1/r$ and $m_{ij}(\lambda_1) = -1/r$

(to see this, use again (3) with $k = \mathrm{dist}(i,j) = 1$). Hence, the number of (i,j)-walks of length $l \geq 0$ is now

$$(\boldsymbol{B}^l)_{ij} = \frac{1}{r}[(r-1)^l - (-1)^l],$$

and the number of vertices of the partite set V_j which are at distance $\leq k$ from $u \in V_i$, $\sigma_k(i,j) := |N_k^+(u) \cap V_j|$, satisfies

$$\begin{aligned} \sigma_k(i,j) \leq \sum_{l=0}^{k} \delta^l (\boldsymbol{B}^l)_{ij} &= \frac{1}{r}\left\{\sum_{l=0}^{k} \delta^l[(r-1)^l - (-1)^l]\right\} \\ &= \frac{1}{r}\left\{\sum_{l=0}^{k} d^l - \sum_{l=0}^{k}(-\delta)^l\right\}. \end{aligned} \tag{8}$$

Let us now consider the behaviour of the second terms in (7) and (8) depending upon the parity of k:

$$\sum_{l=0}^{k}(-\delta)^l = \frac{(-\delta)^{k+1}-1}{-\delta-1} = \begin{cases} -\frac{\delta^{k+1}-1}{\delta+1} & \text{if } k = 2m+1; \\ \frac{\delta^{k+1}+1}{\delta+1} & \text{if } k = 2m. \end{cases}$$

Assume now that $k = 2m+1$ for some integer $m \geq 0$. Then, from the above, we see that $\sigma_k(i,i) \leq \sigma_k(i,j)$ and hence the order of G must satisfy $n \leq r\sigma_k(i,i)$ which yields (a). Similarly, if $k = 2m$ for some integer $m \geq 1$, we have $\sigma_k(i,j) \leq \sigma_k(i,i)$ and (b) comes from $n \leq r\sigma_k(i,j)$. $\square$

Note that the same bounds are obtained when, instead of assuming the regularity condition $\delta = d/(r-1)$, we only suppose that, for every $1 \leq i,j \leq r$, each vertex $u \in V_i$ has *at most* δ outneighbours in V_j (in other words, every vertex has maximum 'interpartite' outdegree δ).

Note also that the above upper bounds correspond to (hypothetical) equipartite digraphs. Thus, assuming that $r \leq d+1$ and for degrees of the form $d = (r-1)\delta$, it makes sense to compare $\overline{M}_r(d,k)$ with $M_r(d,k)$. As shown in the following result, for such values, we can always use the second bound.

Lemma 2.3. *For any given positive integers r, d and k, with $1 < r \leq d+1$ and $d = (r-1)\delta$, we always have $M_r(d,k) \leq \overline{M}_r(d,k)$, and equality occurs if and only if one of the following holds:*

(i) $r = d+1 \quad (\delta = 1)$;

(ii) $k = 1$ *and* $d = 2(r-1) \quad (\delta = 2)$;

(iii) $k = 2m$ *and* $r > M(-\delta, k)$.

Proof. In the extreme case $r = d+1$ ($\delta = 1$), we have that

$$\overline{M}_{d+1}(d,k) = M_{d+1}(d,k) = \begin{cases} M(d,k) & \text{if } \; k = 2m+1, \\ M(d,k)-1 & \text{if } \; k = 2m, \end{cases}$$

so proving (i).

To study the general case $r < d+1$ ($\delta > 1$), note first that, since $\overline{M}_r(d,k) \geq M(d,k) - (r-1)$, we always have

$$\overline{M}_r(d,k) - M_r(d,k) \geq \begin{cases} (r-1)\left(\frac{\delta^{k+1}-1}{\delta+1} - 1\right) & \text{if} \quad k = 2m+1 \\ \frac{\delta^{k+1}+1}{\delta+1} - (r-1) & \text{if} \quad k = 2m. \end{cases}$$

Then, for diameter $k = 2m+1$, it follows that $M_r(d,k) \leq \overline{M}_r(d,k)$ with equality if and only if $\delta^{k+1} - 1 = \delta + 1$; that is, $\delta(\delta^k - 1) = 2$, with the unique solution $k = 1$ and $\delta = 2$ ($d = 2(r-1)$). This proves (ii). The trivial exception corresponds to the complete symmetric digraph K_r^{*2} with two parallel arcs between each (ordered) pair of vertices. For diameter $k = 2m$, the above also gives that $M_r(d,k) < \overline{M}_r(d,k)$, provided that $r \leq M(-\delta,k) = (\delta^{k+1}+1)/(\delta+1)$. Otherwise, if the number of independent vertex sets r is big enough, $r > M(-\delta, 2m)$, we have that $\overline{M}_r(d,2m) = M_r(d,2m)$. This follows from the fact that, since $M_r(d,2m)$ is a multiple of r, we get $M(d,2m) \equiv M(-\delta,2m) \pmod{r}$ which, together with $r > M(-\delta,2m)$, implies that $\lfloor M(d,2m)/r \rfloor = M_r(d,2m)/r$. (For example, if $k = \delta = 2$ then $\overline{M}_r(2r-2,2) = M_r(2r-2,2)$ for any $r \geq 4$.) This proves (iii) and completes the proof of the lemma. □

Note that when $r = d+1$, case (i), the obtained bound coincides with the standard Moore bound (1) if the diameter is odd, whereas for even diameter we obtain the bound for the so-called almost Moore digraphs [3, 20, 17]. This is because the value $r = d+1$ gives (almost) no restriction in the deduction of the upper bound for the number of vertices. Two other particular cases of Proposition 2.2 which are worth mentioning are the following:

(iv) In the bipartite case, $r = 2$, we get $\delta = d$ and the bounds of the above proposition become

$$M_2(d,k) = \frac{d^{k+1}-1}{d-1} - \frac{d^{k+1}-1}{d+1} = 2\frac{d^{k+1}-1}{d^2-1} \tag{9}$$

when the diameter is odd, and

$$M_2(d,k) := \frac{d^{k+1}-1}{d-1} - \frac{d^{k+1}+1}{d+1} = 2\frac{d^{k+1}-d}{d^2-1} \tag{10}$$

for even diameter. This case was studied by the first author and Yebra in [14], where they proved that the above Moore-like bounds can be (and in fact are) attained only when $2 \leq k \leq 4$.

(v) When $k = 2$ we get

$$M_r(d,2) = \left(1 - \frac{1}{(r-1)^2}\right)d^2 + \frac{r}{r-1}d = r\delta[(r-2)\delta + 1]. \tag{11}$$

In particular, when r attains its maximum value $r = d+1$, case (i), we get $M_{d+1}(d,2) = d(d+1)$, which, as we will see later, is attained by the Kautz digraphs.

3 Multipartite Moore digraphs: weakly distance-regularity

Let G be an r-partite δ-equioutregular digraph with outdegree $d = \delta(r-1)$ and diameter k. When its order n attains the Moore-like bound $M_r(d,k)$, given by Proposition 2.2, G is referred to as an *r-partite Moore digraph* (or, generically, *multipartite Moore digraph*). From the proof of Proposition 2.2, the following 'characterization' is derived.

Lemma 3.1. *Let G be an r-partite δ-equioutregular digraph with diameter k. When k is odd [respectively, k is even], G is a multipartite Moore digraph if and only if for any two vertices u and v, belonging to the same partite set [respectively, different partite set], there is a unique walk from u to v of length $\leq k$.*

Thus, in a multipartite Moore digraph with diameter $k = 2m+1$ [respectively, $k = 2m$], there is an invariance of the number of walks between vertices at the same distance, whenever the extra condition of belonging to the same partite set [respectively, different partite set] is added. Such an invariance, which is fulfilled by other Moore-type digraphs (see [9]), leads to the definition of weakly distance-regularity, a concept introduced by Comellas *et al.* in [8]. Formally, a digraph G of diameter k is *weakly distance-regular* if, for each non-negative integer $l \leq k$, the number $a_{uv}^{(l)}$ of walks of length l from vertex u to vertex v depends only on their distance $\mathrm{dist}(u,v) = i$ ($a_i^{(l)} := a_{uv}^{(l)}$). This is equivalent to saying that the distance matrix $\boldsymbol{A}_i$ is a polynomial of degree i in the adjacency matrix $\boldsymbol{A}$; that is, $\boldsymbol{A}_i = p_i(\boldsymbol{A})$, for each $i = 0,1,\ldots,k$, where $p_i \in \mathbb{Q}[x]$ (see [8]). Such polynomials $\{p_i\}_{i=0}^k$ are referred to as the *distance polynomials* of G.

Here we will determine when an r-partite Moore digraph is weakly distance-regular. We will restrict our attention to the case $r > 2$, since Moore bipartite digraphs ($r = 2$) are already known to be weakly distance-regular (see [9]).

Proposition 3.2. *Let G be an r-partite Moore digraph, $r > 2$, with interpartite outdegree δ and diameter k. Then G is weakly distance-regular if and only if one of the following conditions holds:*

(i) $k = 2m+1$ and $\delta = 1$ (G is a Moore digraph);

(ii) $k = 2m$ and each vertex of G is contained in exactly $\epsilon := (\delta^{k+1}+1)/(\delta+1)$ cycles of length k. In such a case, the distance polynomials of G are $p_i = x^i$, $i = 0,1,\ldots,k-1$, and $p_k = x^k - \epsilon$.

Proof. Let $G = (V_1 \cup V_2 \cup \cdots \cup V_r, E)$ be an r-partite Moore digraph with diameter k and let $u \in V_i$ be a vertex of G. We can partition the vertex set $V(G)$ according to the distance from u; that is, $V(G) = \bigcup_{l=0}^k \Gamma_l^+(u)$. Since $r > 2$, $\Gamma_l^+(u)$ contains at least one vertex of each (independent) set V_j, for each $1 \leq j \leq r$ and $2 \leq l \leq k$. So, if we assume that G is weakly distance-regular then, applying Lema 3.1, we deduce that there is a unique walk of length $\leq k$ from u to any vertex v at distance $\mathrm{dist}(u,v) \geq 2$. Moreover, there is also a unique walk of length $\leq k-1$ from u to any vertex $v \in \Gamma_0^+(v) \cup \Gamma_1^+(v)$. Thus, for $k = 2m+1$ [respectively, $k = 2m$] it only remains to determine the number $a_{uv}^{(k)} = a_1^{(k)}$ [respectively, $a_{uu}^{(k)} = a_0^{(k)}$] of $u \to v$ walks [respectively, $u \to u$ (closed) walks] of length k, where $\mathrm{dist}(u,v) = 1$ [respectively, $u = v$]. Now, let us distinguish two cases according to the parity of the diameter k.

For $k = 2m$, since each vertex $v \neq u$ is reached from u in exactly one way in a number of steps $\leq k$, the number of arcs incident from vertices of $\Gamma_{k-1}^+(u)$ to u coincides with the 'defect' of G; that is, $a_0^{(k)} = M(\delta(r-1),k) - M_r(\delta(r-1),k) = (\delta^{k+1}+1)/(\delta+1)$ (the 'defect' measures how far is the order of the digraph from the Moore bound). Note that $a_0^{(k)}$ represents the number of cycles of length k through any given vertex, since there are no cycles of length $< k$.

Analogously, for $k = 2m+1$ we have

$$a_1^{(k)}\delta(r-1) = M(\delta(r-1),k) - M_r(\delta(r-1),k) = (r-1)\frac{\delta^{k+1}-1}{\delta+1}, \qquad (12)$$

since each vertex of $\Gamma_1^+(u)$ ($|\Gamma_1^+(u)| = \delta(r-1)$) must be incident from $a_1^{(k)}$ vertices of $\Gamma_{k-1}^+(u)$ and the total number of these arcs must be equal to the 'defect' of G. From (12), we derive that δ must divide $(\delta^{k+1}-1)/(\delta+1) = \delta^k - \delta^{k-1} + \cdots + \delta - 1$, which is impossible unless $\delta = 1$.

Conversely, if G is a multipartite Moore digraph satisfying condition (ii) then its adjacency matrix $\boldsymbol{A}$ fulfills the equation

$$\boldsymbol{A}^k + \boldsymbol{A}^{k-1} + \cdots + \boldsymbol{A} + (1-\epsilon)\boldsymbol{I} = \boldsymbol{J},$$

where $\boldsymbol{J}$ denotes the all-1 matrix, $k = 2m$ and $\epsilon = (\delta^{k+1}+1)/(\delta+1)$. This is equivalent to saying that the distance-i matrix $\boldsymbol{A}_i$ of G is

$$\boldsymbol{A}_i = \begin{cases} \boldsymbol{A}^i, & \text{if} \quad 0 \leq i \leq k-1 \\ \boldsymbol{A}^k - \epsilon\boldsymbol{I}, & \text{if} \quad i = k, \end{cases}$$

for each $i = 0,1,\ldots,k$. Hence, G is weakly distance-regular and its distance polynomials are $p_i = x^i$, $i = 0,1,\ldots,k-1$, and $p_k = x^k - \epsilon$. Finally, if G is a multipartite Moore digraph with odd diameter and interpartite outdegree $\delta = 1$ then its distance-matrices are $\boldsymbol{A}_i = \boldsymbol{A}^i$, $i = 0,1,\ldots,k$, which means that G is weakly distance-regular. □

4 Some existence conditions

4.1 Weakly distance-regular case

The assumption of being a weakly distance-regular digraph allows us to compute its spectrum from a 'small' matrix, such as the intersection matrix (see [8]). Then, since the eigenvalue multiplicities must be integers, we can derive some necessary conditions about the existence of such a digraph.

First, let us recall the main results on the spectrum of a weakly distance-regular digraph (see [8]). Given a weakly distance-regular digraph G of degree d and diameter k, its distance polynomials $\{p_i\}_{i=0}^k$ satisfy the recurrence relation

$$p_i x = \sum_{j=0}^{i+1} p_{i1}^j p_j \quad (0 \le i \le k-1),$$

where the numbers $p_{i1}^j = |\Gamma_i^+(u) \cap \Gamma_1^-(v)|$, with $\mathrm{dist}(u,v) = j$, are known as the *intersection numbers* of G. Since G is regular, $p_{k1}^j = d - \sum_{i=0}^{k-1} p_{i1}^j$ for each $j = 0, 1, \ldots, k$. The $(k+1) \times (k+1)$ matrix $\boldsymbol{B}$ whose entries are the intersection numbers p_{i1}^j, $(\boldsymbol{B})_{ij} = p_{i1}^j$, is referred to as the *intersection* or *recurrence matrix* of G. We shall say that a vector $\boldsymbol{u}$ is *standard* if its first component is $(\boldsymbol{u})_0 = 1$.

Proposition 4.1 ([8]). *Let G be a weakly distance-regular digraph of degree d, diameter k, order n, and distance polynomials $\{p_i\}_{i=0}^k$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be, respectively, its adjacency and intersection matrices. Then the following statements hold:*

(i) The minimum polynomials of $\boldsymbol{A}$ and $\boldsymbol{B}$ coincide with the characteristic polynomial of $\boldsymbol{B}$ which is

$$\det(x\boldsymbol{I} - \boldsymbol{B}) = \tfrac{1}{\alpha_k^k}(x-d) \textstyle\sum_{i=0}^k p_i,$$

where α_k^k is the leading coefficient of p_k.

(ii) If λ_i is an eigenvalue of $\boldsymbol{B}$ then $\boldsymbol{v}_i = (p_0(\lambda_i), p_1(\lambda_i), \ldots, p_k(\lambda_i))^\top$ is a (right) standard eigenvector of $\boldsymbol{B}$. Moreover, if λ_i has a left standard eigenvector $\boldsymbol{u}_i^\top$ then the multiplicity of λ_i, as an eigenvalue of G, is

$$m(\lambda_i) = \frac{n}{\boldsymbol{u}_i^\top \boldsymbol{v}_i}. \tag{13}$$

Next we apply these results to obtain necessary conditions for the existence of a multipartite Moore digraph G of diameter $k = 2m$ and interpartite outdegree $\delta > 1$ (assuming that G is weakly distance-regular). We will treat separately the case $\delta = 1$ which corresponds to almost Moore digraphs.

Proposition 4.2. *Let G be an r-partite Moore digraph with interpartite outdegree $\delta > 1$ and diameter $k = 2m$. If G is weakly distance-regular then $r > (\delta^k + \delta)(\delta + 1)$ and the number $\sum_{i=0}^{k-1}(-1)^{i+1}(i+1)\delta^i$ divides*

$$r\left(\sum_{i=0}^{k-2} \delta^i \sum_{j=0}^{i} \binom{i+2}{j+2} (-1)^{i-j} r^j\right)\left(1 + r\delta \sum_{i=0}^{k-1} (-1)^{i+1} \delta^i\right).$$

Proof. Let us assume that G is weakly distance-regular. From Proposition 3.2, the distance polynomials of G are $p_i = x^i$, $i = 0, 1, \ldots, k-1$ and $p_k = x^k - \epsilon$, where

$\epsilon = (\delta^{k+1}+1)/(\delta+1)$. Taking into account that G has degree $d = \delta(r-1)$, its intersection matrix $\boldsymbol{B} = (p_{i1}^{j})$ is

$$\boldsymbol{B} = \left(\begin{array}{cccccc} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ . & . & . & . & & \\ . & . & . & . & 1 & \\ \epsilon & 0 & 0 & . & 0 & 1 \\ d-\epsilon & d-1 & d-1 & . & d-1 & d-1 \end{array}\right).$$

Then, from Proposition 4.1, the minimum polynomial of the adjacency matrix $\boldsymbol{A}$ of G coincides with the characteristic polynomial of $\boldsymbol{B}$, which is

$$\begin{aligned} \det(x\boldsymbol{I}-\boldsymbol{B}) &= (x-d)\sum_{i=0}^{k} p_i = (x-d)(x^k + x^{k-1} + \cdots + x + 1 - \epsilon) \\ &= (x-d)(x+\delta)\sum_{i=0}^{k-1}\left(\sum_{j=0}^{k-1-i}(-1)^j\delta^j\right)x^i, \end{aligned}$$

since $\epsilon = \sum_{i=0}^{k}(-1)^i\delta^i$.

The multiplicity $m(-\delta)$ of $-\delta$ can be computed as

$$m(-\delta) = \frac{n}{\boldsymbol{u}_{-\delta}^{\top}\boldsymbol{v}_{-\delta}},$$

where n is the order of G and $\boldsymbol{v}_{-\delta} = (1, -\delta, \delta^2, \ldots, -\delta^{k-1}, \delta^k - \epsilon)^{\top}$, $\boldsymbol{u}_{-\delta}^{\top} = (1, x_1, \ldots, x_D)$ represent the left and right standard eigenvectors of $\boldsymbol{B}$ corresponding to $-\delta$, respectively. The condition $\boldsymbol{u}_{-\delta}^{\top}\boldsymbol{B} = -\delta\boldsymbol{u}_{-\delta}^{\top}$ can be expressed as the linear system

$$\left.\begin{array}{rcl} \epsilon x_{k-1} + (d-\epsilon)x_k &=& -\delta \\ x_{k-1} + (d+\delta-1)x_k &=& 0 \end{array}\right\} \tag{14}$$

together with the recurrence relation

$$x_i = -\frac{1}{\delta}(x_{i-1} + (d-1)x_k),\ \ i = 1, 2, \ldots, k-1,\ \text{ and } x_0 := 1. \tag{15}$$

Solving (14), we get

$$x_k = \frac{1}{r(\epsilon-1)+1}.$$

Then, applying (15), we have

$$(-\delta)^i x_i = 1 + (d-1)x_k\sum_{j=0}^{i-1}(-\delta)^j,\ \ i = 1, 2, \ldots, k-1.$$

Therefore,

$$\begin{aligned}
\boldsymbol{u}_{-\delta}^{\top}\boldsymbol{v}_{\delta} &= 1+\sum_{i=1}^{k-1}\left[1+(d-1)x_k\sum_{j=0}^{i-1}(-\delta)^j\right]-x_k\sum_{j=0}^{k-1}(-\delta)^j \\
&= k+x_k\left[(d-1)\sum_{i=1}^{k-1}\frac{1-(-\delta)^i}{\delta+1}-\frac{1-\delta^k}{\delta+1}\right] \\
&= k+x_k\frac{\delta^{k+1}+d\delta^k+(kd-k-1)\delta+kd-k-d}{(\delta+1)^2} \\
&= k+x_k\sum_{i=1}^{k}\delta^{k-i}(-1)^i\left[(i-1)d-i\right] \\
&= k+x_k\left[\sum_{i=1}^{k-1}(-1)^{i-1}ir\delta^{k-i}-k\right].
\end{aligned}$$

Taking into account that $x_k=\frac{1}{r(\epsilon-1)+1}$ and $\epsilon=\sum_{i=0}^{k}(-\delta)^i$, we get

$$\boldsymbol{u}_{-\delta}^{\top}\boldsymbol{v}_{-\delta}=\frac{\delta r\displaystyle\sum_{i=1}^{k}(-1)^i i\delta^{i-1}}{\delta r\displaystyle\sum_{i=1}^{k}(-1)^i\delta^{i-1}+1}.$$

Let us denote

$$a(\delta):=\sum_{i=1}^{k}(-1)^i i\delta^{i-1}\ \text{and}\ b(\delta):=\sum_{i=1}^{k}(-1)^i\delta^{i-1}.$$

Then

$$m(-\delta)=\frac{\frac{n}{\delta r}\left[\delta r b(\delta)+1\right]}{a(\delta)},$$

where $n=\sum_{i=1}^{k}(\delta^i(r-1)^i-(-\delta)^i)$. It can be checked that

$$\frac{n}{r\delta}=\sum_{i=0}^{k-1}\delta^i\sum_{j=0}^{i}\binom{i+1}{j+1}(-1)^{i-j}r^j.$$

We can express

$$\frac{n}{r\delta}=-a(\delta)+\delta r c(\delta,r),\ \text{where}\ c(\delta,r):=\sum_{i=0}^{k-2}\delta^i\sum_{j=0}^{i}\binom{i+2}{j+2}(-1)^{i-j}r^j.$$

Thus

$$m(-\delta)=\left(-1+\frac{\delta r c(\delta,r)}{a(\delta)}\right)(\delta r b(\delta)+1).$$

Since $m(-\delta)$ is an integer and $\gcd(\delta, a(\delta)) = 1$, we conclude that

$$a(\delta) \mid rc(\delta, r)[\delta r b(\delta) + 1].$$

Finally, since $d = \delta(r-1) \geq \epsilon$ and $\epsilon = \frac{\delta^{k+1}+1}{\delta+1}$, we get

$$r \geq \frac{\delta^k + \delta}{\delta + 1} + \frac{1}{\delta} > \frac{\delta^k + \delta}{\delta + 1}.$$

□

We remark that the diameter 2 weakly distance-regular digraphs are the same as the 'directed strongly regular graphs' introduced by Duval [10]. Duval defined a (n, d, μ, λ, t)-*graph* to be a directed graph with order n whose adjacency matrix $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}^2 + (\mu - \lambda)\boldsymbol{A} - (t - \mu)\boldsymbol{I} = \mu \boldsymbol{J} \quad \text{and} \quad \boldsymbol{AJ} = \boldsymbol{JA} = d\boldsymbol{J}.$$

Within this framework, we are interested in finding r-partite Moore digraphs with interpartite outdegree δ such that the digraphs are $(n, d, 1, 0, t)$-graphs, where $n = r\delta[(r-2)\delta + 1]$, $d = \delta(r-1)$ and $t = \delta^2 - \delta + 1$. We also point out that $(n, d, 1, 0, t)$-graphs are the same as the *mixed Moore graphs* (with undirected degree t), introduced by Bosák [5].

Corollary 4.3. *Let G be an r-partite Moore digraph with interpartite outdegree $\delta > 1$ and diameter $k = 2$. If G is weakly distance-regular then $r > \delta$ and*

$$(2\delta - 1) \mid r[r\delta(\delta - 1) + 1]. \tag{16}$$

In particular, if $2\delta - 1$ is a prime number then $r \equiv 0, 4 \pmod{2\delta - 1}$.

Proof. Applying Proposition 4.2, we have $r > \frac{\delta^2+\delta}{\delta+1} = \delta$ and

$$r[r\delta(\delta - 1) + 1] \equiv 0 \pmod{2\delta - 1}.$$

Thus, if $2\delta - 1$ is a prime number then $r \equiv 0 \pmod{2\delta - 1}$ or $r\delta(\delta - 1) \equiv 2(\delta - 1) \pmod{2\delta - 1}$. Since $\gcd(\delta - 1, 2\delta - 1) = 1$, the latter congruence is equivalent to saying that $r\delta \equiv 2 \pmod{2\delta - 1}$, which has as the solution $r \equiv 4 \pmod{2\delta - 1}$. □

The list of pairs (δ, r) satisfying the divisibility condition (16) and associated with a Moore multipartite digraph of diameter 2 and order $r\delta[(r-2)\delta+1] \leq 100$ are shown in Table 1.

As can be seen in Figure 1, Bosák graph is 3-partite 2-equioutregular. Since each of its vertices is contained in a 3-cycle and has undirected degree 3, Bosák graph cannot be equiregular.

4.2 Case $\delta = 1$ $(r = d + 1)$

Let us now consider case (i) where $r = d + 1$. We see, by (i) and the results in [22], that, when the diameter is odd, $(d+1)$-partite Moore digraphs exist only in the

δ	r	$d=\delta(r-1)$	$t=\delta^2-\delta+1$	$n=r\delta((r-2)\delta+1)$	Example
2	3	4	3	18	Bosák graph (unique solution; see [21])
2	4	6	3	40	unknown
3	4	9	7	84	unknown

Table 1: Feasible parameters of a multipartite Moore digraph with diameter 2 and order ≤ 100 in the weakly distance-regular case.

trivial case $k=1$ (the complete symmetric digraph, here denoted by K_r^1). Moreover, for even diameter, the obtained bound is the same as for almost Moore (standard) digraphs, namely $M(d,k)-1$.

We recall that every almost Moore digraph G of diameter k has the characteristic property that for each vertex $v \in V(G)$ there exists exactly one vertex, denoted by $r(v)$ and called the *repeat* of v, such that there are exactly two $v \to r(v)$ walks of length at most k (one of them must be of length k). If $r(v)=v$, which means that v is contained in exactly one k-cycle, v is called a *selfrepeat* of G. Moreover, if $w \neq r(v)$ then there is a unique $v \to w$ walk of length $\leq k$. From the (di)regularity of an almost Moore digraph, proved by Miller *et al.* [20], it follows that the map r, which assigns to each vertex $v \in V(G)$ the vertex $r(v)$, is an automorphism of G (see [4]). Seeing it as a permutation, r has a *cycle structure* which corresponds to its unique decomposition into disjoint cycles. Such cycles are called *permutation cycles* of G. The number of permutation cycles of G of each length $i \leq n = M(d,k)-1$ is denoted by m_i and the vector $(m_1,\ldots,m_n)$, which represents a partition of the n vertices of G into m_i i-sets of vertices, for $i=1,\ldots,n$, is referred to as the *permutation cycle structure* of G.

The following lemma provides a necessary condition for the existence of a Moore multipartite digraph, with interpartite outdegree $\delta=1$ and diameter $k=2m$, in terms of its permutation cycle structure.

Lemma 4.4. *Let G be an almost Moore digraph of degree $d>1$ and diameter $k=2m$. If G is $(d+1)$-partite and (1-)equioutregular then all permutation cycles of G have the same length l. Moreover, all vertices of a permutation cycle of G belong to the same independent set. In particular, l divides $\frac{d^{2m+1}-d}{d^2-1}$.*

Proof. Let $V=V_1\cup V_2\cup\cdots\cup V_{d+1}$ be a partition of the set of vertices of G such that any vertex of V_i is adjacent to exactly one vertex of each set V_j, $j\neq i$, for $i=1,2,\ldots,d+1$. First, we note that each vertex v and its repeat $r(v)$ belong to the same independent vertex set, since from the deduction of the bound (6) we know that there is a unique walk of length $\leq k$ between any two vertices of distinct independent sets. Therefore, all vertices of a permutation cycle of G belong to the same partite set. Let $(v, r(v),\ldots,r^{l-1}(v))$ be a permutation cycle of minimum length (l) and let $\Gamma_1^+(v)=\{w_1,w_2,\ldots,w_d\}$. Since r is an automorphism of G, it follows that $\Gamma_1^+(v)=\{r^l(w_1),r^l(w_2),\ldots,r^l(w_d)\}$. Then, taking into account that G is (1-)equioutregular, we have that $r^l(w_i) = w_i$, whence we deduce that the order of w_i is equal to the minimum l, for each $i = 1, \dots, d$. Hence, all vertices of $\Gamma_1^+(v)$ have the same order, which implies that all permutation cycles of G have the same length l. Clearly, l divides the number of vertices of each set V_i, which is $(M(d, 2m) - 1)/(d + 1) = \sum_{i=1}^{m} d^{2i-1}$. □

From the previous lemma and taking into account the nonexistence of almost Moore digraphs of diameter $k > 2$ with all selfrepeats (see [3]), we conclude that a Moore $(d+1)$-partite (1-)equioutregular digraph G with diameter $k = 2m > 2$ does not contain any k-cycle. Therefore, using properties of graphical cycles of an almost Moore digraph (see [17]), we can deduce that there is a partition of the set of arcs of G into $(k+1)$-cycles, which implies that $(k+1)$ divides $d(M(d, 2m) - 1)$.

Proposition 4.5. *Almost Moore digraphs of degree $d > 1$ and diameter $k > 1$ with all permutation cycles of order two do not exist.*

Proof. Let us assume that there is an almost Moore digraph G with degree $d > 1$, diameter $k > 1$ and permutation cycle structure $m_2 = n/2$, where $n = d + d^2 + \cdots + d^k$. Let $\boldsymbol{A}$ be the adjacency matrix of G.

From [17, Proposition 1], we obtain the factorization in $\mathbb{Q}[x]$ of the characteristic polynomial of G, which is

$$\phi(G, x) = (x - d)x^{a_1} \prod_{\substack{l|k \\ l \neq 1}} [\Phi_l(x)]^{a_l} \, (x^k + \cdots + x + 2)^{\frac{n}{2k}},$$

where $\Phi_l(x)$ denotes the lth cyclotomic polynomial and the (unknown) multiplicities a_l are non-negative integers.

Furthermore, since G does not contain any closed walk of length $< k$, we have $\operatorname{tr} \boldsymbol{A} = \operatorname{tr} \boldsymbol{A}^{k-1} = 0$. Such conditions can be expressed in terms of the eigenvalues of $\boldsymbol{A}$ as follows.

$$\begin{aligned}
\operatorname{tr} \boldsymbol{A} \quad &= \quad d + \sum_{\substack{l|k \\ l \neq 1}} a_l \sum_{\substack{\gcd(i,l)=1 \\ 1 \le i \le l}} \xi_l^i + \frac{n}{2k} \sum_{i=1}^{k} \lambda_i \\
\operatorname{tr} \boldsymbol{A}^{k-1} \quad &= \quad d^{k-1} + \sum_{\substack{l|k \\ l \neq 1}} a_l \sum_{\substack{\gcd(i,l)=1 \\ 1 \le i \le l}} \xi_l^{i(k-1)} + \frac{n}{2k} \sum_{i=1}^{k} \lambda_i^{k-1},
\end{aligned}$$

where ξ_l is an lth primitive root of unity and $\lambda_1, \lambda_2 \dots, \lambda_k$ are the distinct zeros of $x^k + \cdots + x + 2$.

Note that, for each $l|k$, we have

$$\sum_{\substack{\gcd(i,l)=1 \\ 1 \le i \le l}} \xi_l^i = \sum_{\substack{\gcd(i,l)=1 \\ 1 \le i \le l}} (\xi_l^{k-1})^i,$$

since $\gcd(k-1,l)=\gcd(k-1,k)=1$ and, consequently, ξ_l^{k-1} is also an lth primitive root of unity.

In order to compute $S_h=\sum_{i=1}^{k}\lambda_i^h$ $(h=1,k-1)$, we will use Newton's formulas, which express such sums in terms of the elementary symmetric functions

$$\Sigma_h=\sum_{1\le j_1<j_2<\cdots<j_h\le k}\lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_h},$$

as

$$S_h=S_{h-1}\Sigma_1-S_{h-2}\Sigma_2+\cdots+(-1)^{h-2}S_1\Sigma_{h-1}+(-1)^{h-1}h\Sigma_h. \tag{17}$$

Taking into account that $\lambda_1,\lambda_2,\cdots,\lambda_k$ are the roots of the polynomial $x^k+x^{k-1}+\cdots+x+2$, we have

$$\Sigma_h=\begin{cases}(-1)^h, & \text{if } \ 1\le h\le k-1\\ (-1)^k2, & \text{if } \ h=k.\end{cases} \tag{18}$$

Using an inductive argument we can prove that $S_h=-1$ for $h=1,\ldots,k-1$. First, $S_1=\Sigma_1=-1$. Now, let us assume that $S_i=-1$ for each $i\le h-1$. From (17) and (18), we have

$$S_h=-(S_{h-1}+S_{h-2}+\cdots+S_1)-h=-1.$$

Therefore, the condition $\operatorname{tr}\boldsymbol{A}=\operatorname{tr}\boldsymbol{A}^{k-1}$ implies that

$$d=d^{k-1},$$

which is impossible unless $k=2$. From the classification of almost Moore digraphs of diameter two, given in [18], we conclude that there does not exist an almost Moore digraph with all permutation cycles of order two. $\square$

In the case of diameter two, from the work of various authors (see e.g., [16, 2, 17]), it is known that there is a unique almost Moore digraph, namely the Kautz digraph of diameter two, $K(d,2)$, apart from the particular case of degree $d=2$ for which there are two more digraphs. The Kautz digraph $K(d,2)$ coincides with the line digraph LK_{d+1}^1 of the complete symmetric digraph K_{d+1}^1 (see [15]). We recall that, given a digraph G, its line digraph LG has as a set of vertices $V(LG)=E(G)$ and its adjacencies are defined as follows

$$(u_1v_1,u_2v_2)\in E(LG)\iff v_1=u_2,\quad \forall u_1v_1,u_2v_2\in E(G).$$

Remark 4.6. *If $G=(V_1\cup V_2\cup\cdots\cup V_r,E)$ is an r-partite and δ-equioutregular digraph then its line digraph LG is also r-partite and δ-equioutregular, since we can consider the 'inherited' vertex partition $V(LG)=\cup_{i=1}^r\{(u,v)\in E,\ v\in V_i\}$ that preserves the equioutregularity. Moreover, if $\delta=1$ and $r=d+1>2$ then LG cannot be equiregular, since the above vertex partition of LG, which is the unique one (modulo rearragements of its partite sets) that provides the equioutregularity, it does not fulfill the equiinregularity condition.*

Indeed, the line digraph LK^1_{d+1} has $d(d{+}1)$ vertices and is $(d{+}1)$-partite $(1-)$equioutregular [$(1-)$equiinregular] with independent vertex sets $V_i = \{ji : j \neq i\}$ [$V_i = \{ij : j \neq i\}$], $1 \leq i \leq d+1$. Nevertheless, $K(d,2)$ is not equiregular. We remark that the other two almost Moore digraphs of diameter $k = 2$ and degree $d = 2$, given in [2], are $(d+1)$-partite but not even equioutregular.

In general, the following result can be of some help in proving existence results of a multipartite (di)regular digraph G by using information on its spectrum.

Lemma 4.7. *Let G be an r-partite δ-equiregular Moore digraph. Then G has the eigenvalues $d = \delta(r-1)$ and $-\delta$ with multiplicities 1 and $r-1$, respectively. Moreover, any λ-eigenvector with $\lambda \neq d, -\delta$ is orthogonal to the eigenspaces corresponding to d and $-\delta$.*

Proof. Let G have adjacency matrix $\boldsymbol{A}$ with blocks $\boldsymbol{A}_{ij}$ representing the adjacencies from V_i to V_j, $1 \leq i, j \leq r$. Since G is connected and δ-equiregular, it has $d = (r-1)\delta$ as a simple eigenvalue with eigenvector $\boldsymbol{j}$. Moreover, since G admits the regular (or equitable) partition $V_1 \cup V_2 \cup \cdots \cup V_r$, with quotient matrix $\delta \boldsymbol{A}(K_r)$, G has also the eigenvalue $-\delta$ with multiplicity $r-1$ and eigenvectors

$$\boldsymbol{y} = (\boldsymbol{y}_1|\boldsymbol{y}_2|\boldsymbol{y}_3|\cdots|\boldsymbol{y}_r) = (\boldsymbol{j}|\omega^i\boldsymbol{j}|\omega^{2i}\boldsymbol{j}|\ldots|\omega^{(r-1)i}\boldsymbol{j})^\top \qquad (1 \leq i \leq r),$$

where ω is a primitive r-th root of 1, say $\omega := e^{j\frac{2\pi}{r}}$. [We recall that K_r has eigenvalues $r-1$ and -1 (with multiplicity $r-1$), with corresponding orthogonal eigenvectors $\boldsymbol{j} \in \mathbb{R}^r$ and $\boldsymbol{\phi}_i = (1, \omega^i, \omega^{2i}, \ldots, \omega^{(r-1)i})^\top$, $1 \leq i \leq r-1$.]

Now, from the above, it suffices to prove that, if $\boldsymbol{x} = (\boldsymbol{x}_1|\boldsymbol{x}_2|\cdots|\boldsymbol{x}_r)$ is a λ-eigenvector (with $\lambda \neq d, -\delta$) then $\langle \boldsymbol{x}_i, \boldsymbol{j}\rangle = 0$, for any $1 \leq i \leq r$. With this aim, note first that, since G_{ij} (the subdigraph with vertex set $V_i \cup V_j$ and arcs from V_i to V_j) is δ-inregular, we have

$$\langle \boldsymbol{A}_{ij}\boldsymbol{x}_j, \boldsymbol{j}\rangle = \langle \boldsymbol{x}_j, \boldsymbol{A}_{ij}^\top \boldsymbol{j}\rangle = \delta\langle \boldsymbol{x}_j, \boldsymbol{j}\rangle \qquad (j \neq i). \tag{19}$$

From $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, we get the r equations

$$\sum_{j=1}^{r} \boldsymbol{A}_{ij}\boldsymbol{x}_j = \lambda\boldsymbol{x}_i \qquad (1 \leq i \leq r)$$

whence, multiplying both terms by $\boldsymbol{j} \in \mathbb{R}^r$ and using (19),

$$\sum_{j=1, j\neq i}^{r} \delta\langle \boldsymbol{x}_j, \boldsymbol{j}\rangle = \lambda\langle \boldsymbol{x}_i, \boldsymbol{j}\rangle \qquad (1 \leq i \leq r)$$

The above r equations constitute an homogeneous linear system with unknowns $\langle \boldsymbol{x}_i, \boldsymbol{j}\rangle$ and coefficient matrix $\delta\boldsymbol{A}(K_r) - \lambda\boldsymbol{I}$. Consequently, if λ is not a zero of $\det(\delta\boldsymbol{A}(K_r) - \lambda\boldsymbol{I})$, that is $\lambda \neq (r-1)\delta, -\delta$, such a system has only the trivial solution $\langle \boldsymbol{x}_i, \boldsymbol{j}\rangle = 0$, $1 \leq i \leq r$, as claimed. □

Note that, if we require G to be only (δ-)equioutregular then $-\delta$ is also an eigenvalue of G with algebraic multiplicity at least $r-1$. However, the orthogonality property of the eigenspaces requires the equiregularity condition.

5 Some constructions

Of course, the line digraph operation behaves well, giving an r-partite digraph when applied to an r-partite digraph. So the most difficult task is to construct the small (not the trivial smallest) dense multipartite digraphs.

For $r = 2$, see [14, 13].

For $k = 2$ we can use the following simple construction which provides not too bad results. Let K_r^δ denote the complete symmetric digraph with δ parallel arcs between any two vertices. Then its line digraph LK_r^δ is also r-partite, it has diameter two, and its order is $n = r(r-1)\delta$. Comparing this number with bound (11) in case (v), we see that this construction gives better results when δ is small. More precisely, note that, for large values of r, bound (11) is of the order of δ times the above value of n.

A better construction for $k = 2$ is obtained by joining appropriately $t \geq 3$ copies of the Kautz digraph $K(r-1, 2)$. The proposed digraph $G_{t,r}$ has vertex set

$$V = \{\alpha|ij : \alpha \in \mathbb{Z}_t;\ i, j \in \mathbb{Z}_r, i \neq j\}$$

and adjacencies

$$\alpha|ij \to \alpha|jk \quad (k \neq j) \qquad \text{and} \qquad \alpha|ij \to \beta|kj \quad (\beta \neq \alpha, k \neq i, j).$$

Thus, $G_{t,r}$ is a d-outregular digraph with degree $d = (r-1)+(t-1)(r-2) = rt-2t+1$, and it has $n = tr(r-1)$ vertices. Moreover, we have the following result.

Proposition 5.1. *Given integers $t, r \geq 3$, $G_{t,r}$ is a vertex-transitive, r-partite digraph with diameter $k = 2$.*

Proof. The facts that $G_{t,r}$ is vertex symmetric and r-partite follow directly from its definition. Thus, its partite sets are $V_i = \{\alpha|ij : \alpha \in \mathbb{Z}_t, j \in \mathbb{Z}_r \setminus \{i\}\}$, $0 \leq i \leq r-1$. Moreover, because of symmetry, we only need to show the paths (of length ≤ 2) from vertex $0|01$ to the vertices in another copy, say, those of type $1|ij$. These paths are the following:

$$\begin{array}{lllll}
0|01 & \to & 1|i1\ (i \neq 0,1); & & \\
0|01 & \to & 1|i1\ (i \neq 0,1) & \to & 1|1j\ (j \neq 1); \\
0|01 & \to & 0|1j\ (j \neq 1) & \to & 1|ij\ (j \neq 1, i \neq 1, j); \\
0|01 & \to & 2|21 & \to & 1|01. \quad \square
\end{array}$$

Let us now consider some particular cases of the above construction.

(a) When $t = r$ we get a $(r-1)^2$-regular digraph on $n = r^2(r-1) = (\sqrt{d}+1)^2\sqrt{d}$ vertices, which is of the order of $d^{\frac{3}{2}}$. Note that the corresponding Moore bound (6) would be

$$M_r(d, 2) = d^2 + \sqrt{d}.$$

(b) When $r = 3$, we get the digraph $G_{t,3} \equiv G(6t, t+1, 2)$, with $n = 6t$ vertices and degree $d = t+1$, whereas in this case the Moore bound (6) gives

$$M_3(t+1, 2) = \frac{3}{2}t(t+1). \tag{20}$$

In particular, if $t = 3$ the digraph $G(18, 4, 2)$ is isomorphic to the Bosák graph, shown in Figure 1 (see [5]), and its order attains the above bound. Note that its line digraph $LG(18, 4, 2)$ is a 3-partite 4-regular digraph with 72 vertices and diameter $k = 3$, only 3 vertices less than the corresponding Moore bound.

(c) When $r = 4$ we get the digraph $G_{t,4} \equiv G(12t, 2t+1, 2)$, ...

Note that, in general, the obtained digraphs are not equiregular, not even equioutregular.

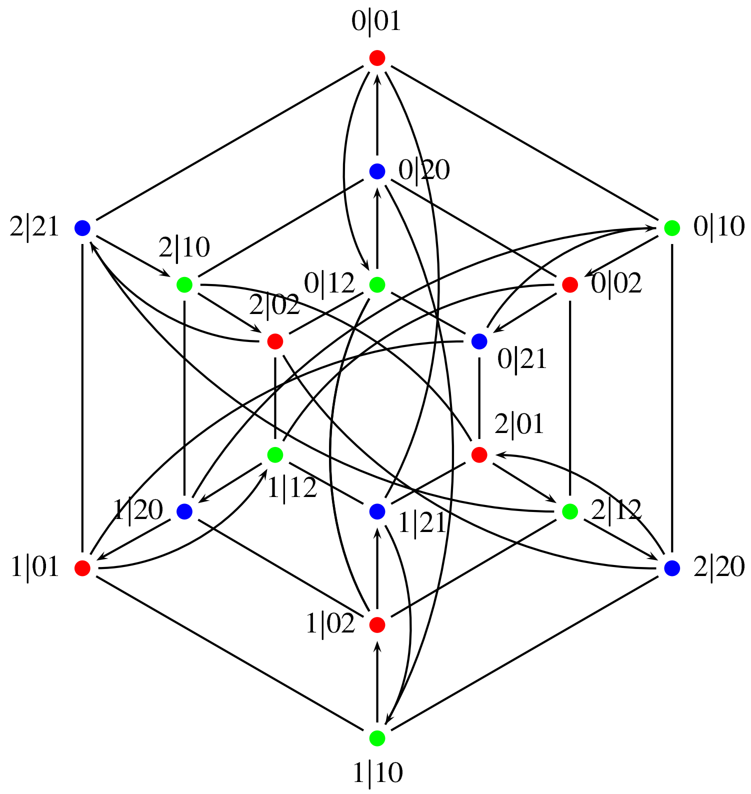

Figure 1: The Bosák graph (see [5]).

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